

# On the Forward Scattering Amplitude of the Virtual Longitudinal Photon at Zero Energy.

B.L.Ioffe

*Institute of Theoretical and Experimental Physics*

*B.Chermushkinskaya 25, 117259, Moscow, Russia*

The theorem is proved, determining the forward scattering amplitude of virtual longitudinal photon at zero energy on any hadronic target in the limit of small photon virtualities  $Q^2$ . The theorem is strict, based only on Lorenz and gauge invariance. No assumptions about the strong interaction, besides the gap in the mass spectrum, are used. Two terms in the expansion over  $Q^2$  are calculated. The estimation is given up to what  $Q^2$  the elastic term can represent the whole amplitude of longitudinal virtual photon-proton scattering.

PACS numbers: 03.80.+r, 11.55.-m, 11.55.Fv., 13.60.-r

The knowledge of the forward scattering amplitude of the virtual longitudinal photon is desirable for many purposes. This amplitude appears in the electron-nucleon scattering, in the approaches, connected with vector dominance model etc. The forward scattering amplitude at zero photon energy arises as a subtraction term in the dispersion relation for this amplitude.

In this paper I prove the theorem, determining the value of this amplitude at zero photon energy at small photon virtualities. This theorem is similar to the famous theorem, proved many years ago by Thirring [1], Kroll and Ruderman [2], Low [3] and Gell-Mann and Goldberger [4], where it was shown that in the case of real (i.e.transverse) photon the scattering amplitude at zero photon energy on any target is given by Thomson formula:  $f_T(0) = -Z^2\alpha/m$  [1,2] and the terms linear in the photon energy for spin 1/2 target are expressed through the target static magnetic moment [3,4]. (Here  $\alpha$  is the fine structure constant,  $Z$  and  $m$  are the target charge and mass.)

The virtual photon forward scattering amplitude  $f(\nu, q^2)$  is the function of two invariants  $\nu = pq$  and  $q^2$ , where  $p$  and  $q$  are the target and photon 4-momenta. I am interested in the case  $\nu = 0$  and  $q^2 < 0$  (metrics:  $q^2 = q_0^2 - \mathbf{q}^2$ ). It is convenient

to go to the Lorenz coordinate system, where  $q_0 = 0$  and the  $z$ -axis is the collision axis. In this coordinate system  $q^2 = -q_z^2$  and the condition  $\nu = 0$  means that  $p_z = 0$  – the coordinate system coincides with the laboratory system. The forward scattering amplitude of longitudinal virtual photon is given by

$$f_L(\nu, q^2) = e_{L\mu} T_{\mu\lambda}(\nu, q^2) e_{L\lambda}, \quad (1)$$

where  $e_{L\mu}$  is the photon longitudinal polarization,

$$T_{\mu\lambda}(\nu, q^2) = ie^2 \int d^4x e^{iqx} \langle p | T \{ j_\mu(x), j_\lambda(0) \} | p \rangle, \quad (2)$$

and  $j_\mu(x)$  is the electromagnetic current. From the condition  $e_L q = 0$  in the chosen coordinate system we have  $e_0 = 1, \mathbf{e} = 0$ . Therefore in this coordinate system

$$f_L(\nu, q^2) = ie^2 \int d^4x e^{-iq_z z} \langle p | T \{ j_0(x), j_0(0) \} | p \rangle, \quad (3)$$

Represent (3) as the sum over the whole set of intermediate states and integrate over the time, using

$$j_0(t, \mathbf{x}) = e^{iHt} j_0(0, \mathbf{x}) e^{-iHt},$$

where  $H$  is the Hamiltonian. At  $\nu = 0$  we have

$$\begin{aligned} f_L(0, q^2) = -e^2 \sum_n \int d^3x e^{-iq_z z} \frac{1}{E_p - E_n} & \left[ \langle p | j_0(0, \mathbf{x}) | n \rangle \langle n | j_0(0) | p \rangle \right. \\ & \left. + \langle p | j_0(0) | n \rangle \langle n | j_0(0, \mathbf{x}) | p \rangle \right], \end{aligned} \quad (4)$$

where  $E_p = m$  and  $E_n$  are the energies of initial and intermediate states,  $m$  is the target mass. Go to the limit  $q_z \rightarrow 0$  and integrate over  $d^3x$  in (4). Then

$$\int d^3x j_0(0, \mathbf{x}) = \mathbf{Z} \quad (5)$$

– the total charge operator. Among the intermediate states in (4) only the target state gives nonvanishing contribution to (4). Its energy is equal to

$$E_n = \sqrt{m^2 + q_z^2} \approx m + \frac{1}{2} \frac{q_z^2}{m} \quad (6)$$

From (4),(5),(6) we get the general formula ( $Q^2 = -q^2 > 0$ )

$$f_L(0, Q^2) = 4\alpha Z^2 \frac{m}{Q^2}, \quad (7)$$

where  $Z$  is the target charge. Remarkable features of eq.7 are the singularity in  $Q^2$  and the sign opposite to the Thomson amplitude.

The derivation of eq.7, is based only on charge conservation. It holds for any strong interacting target and no assumptions about strong interaction, besides the gap in the mass spectrum are used. The latter is necessary, because, if the intermediate states would be degenerate, the proof, in general, would fail. For this reason the case of QED, where such degeneration takes place due to infrared problem, requires a special investigation.

The proof is valid strictly at  $\nu = 0$ . At  $\nu \neq 0$  and  $Q^2 \rightarrow 0$   $f_L(\nu, Q^2) \sim Q^2$ , as could be expected *a priori*. (The formal reason, why the presented above derivation fails at  $\nu \neq 0$ , is that in the chosen coordinate system, the initial state momentum  $p_z = -\nu/q_z$  and goes to infinity at  $Q^2 \rightarrow 0$ ). However, the proof may be generalized to the "scaling case"  $\nu \sim Q^2 \rightarrow 0$  (see below, eq.10).

For spin zero target the next order term in  $Q^2$  can also be calculated from (4). Again, only intermediate state, coinciding with the initial one, gives a nonvanishing contribution in this order. The second order term in  $Q^2$  arises from two sources: 1) from the expansion of the exponent in (4) up to the second order; 2) from the next term in the expansion of the denominator in  $q_z^2$ . As a result we get instead of (7) the following formula, correct up to  $O(Q^2)$  terms:

$$f_L(0, Q^2)_{spin\ 0} = 4Z^2\alpha \frac{m}{Q^2} \left[ 1 + \frac{Q^2}{4m^2} - \frac{1}{3}Q^2 \langle r_E^2 \rangle \right], \quad (8)$$

where  $\langle r_E^2 \rangle$  is the target mean square charge radius

$$\langle r_E^2 \rangle = \langle p | \int d^3x \mathbf{r}^2 j_0(x) | p \rangle / \langle p | \int j_0(x) d^3x | p \rangle \quad (9)$$

Consider now the next order terms in  $Q^2$  for the spin 1/2 target. The situation here is more complicated, because of presence of static magnetic moment. For definiteness consider the most interesting case of virtual photon-proton scattering. It is instructive to calculate the contribution of intermediate proton state to forward

virtual photon-proton scattering amplitude in general case,  $\nu \neq 0$ . The calculation gives:

$$f_L(\nu, Q^2)_{proton} = -\alpha Q^2 m \left[ \frac{1}{4m^4} F_M^2(Q^2) + \frac{1}{\nu^2 - Q^4/4} G_E^2(Q^2) \right], \quad (10)$$

where  $F_M(Q^2)$  is the Pauli magnetic formfactor,

$$F_M(Q^2) = \frac{1}{1 + Q^2/4m^2} \left[ G_M(Q^2) - G_E(Q^2) \right]. \quad (11)$$

$G_E(Q^2)$  and  $G_M(Q^2)$  are Sachs proton electric and magnetic formfactors. At  $\nu = 0$  and  $Q^2 \rightarrow 0$  (10) evidently reduces to (7). From (10) is clear, that the case of  $\nu = 0$  is a special one: at  $\nu = Const \neq 0$  and  $Q^2 \rightarrow 0$   $f_L(\nu, Q^2) \sim Q^2$  and vanishes, as it should be for longitudinal photon scattering amplitude.

Let us now put  $\nu = 0$ . The contribution of the excited intermediate state to  $f_L(0, Q^2)$  may be estimated as

$$f_L(0, Q^2)_{excit.} \sim \frac{\alpha}{m} \frac{Q^2}{M^2 - m^2 + Q^2} G_{pN^*}^2(Q^2) \quad (12)$$

for spin 1/2 state  $N^*$ , where  $G_{pN^*}(Q^2)$  is the  $p \rightarrow N^*$  transition formfactor. (The factor  $Q^2$  comes from gauge invariance). For higher spins the additional factor of  $(Q^2)^n$  appears in the numerator. Since the contributions of excited states are proportional to  $Q^2$  the constant, independent on  $Q^2$  term in the expansion of  $f_L(0, Q^2)$  at small  $Q^2$  arises entirely from the proton intermediate state and can be found from (10). So, generally, for spin 1/2 target we get up to terms of  $O(Q^2)$

$$f_L(0, Q^2)_{spin\ 1/2} = 4Z^2 \frac{\alpha m}{Q^2} \left[ 1 - \frac{Q^2}{2m^2} \mu_a - \frac{1}{3} \langle r_E^2 \rangle Q^2 \right], \quad (13)$$

where  $\mu_a$  is the anomalous magnetic moment. In deriving (13) the connection of Sachs and Pauli formfactors were exploited

$$G_E(Q^2) = F_E(Q^2) - \frac{Q^2}{4m^2} F_M(Q^2), \quad (14)$$

$$\langle r_E^2 \rangle = -6 \frac{dF_E(Q^2)}{dQ^2} \Big|_{Q^2=0}. \quad (15)$$

As can be seen from (10), (14) (as well as from (4))  $f_L(0, Q^2)$  is zero up to terms  $O(Q^2)$  for neutral particles, for example, for neutron.

Let us estimate finally up to what  $Q^2$  the proton intermediate state contribution is dominating in longitudinal virtual photon-proton scattering amplitude at  $\nu = 0$ . Since the transition formfactor  $G_{pN}^*(Q^2)$  in (12) is of the same order or smaller, than  $G_E(Q^2)$  (especially for highly excited states), then the estimation (12) gives that one expect such dominance up to  $Q^2 \sim 0.5\text{GeV}^2$ . This expectation may be checked by calculation of the isobar intermediate state contribution to the forward scattering amplitude of virtual longitudinal photon on the proton at  $\nu = 0$ . It can be shown, that only Coulomb quadrupole  $p \rightarrow \Delta$  transition formfactor  $G_c^*(Q^2)$  contributes here. The calculation gives (in the notation of ref.5):

$$f_L(0, Q^2)_\Delta = 12\alpha M m^2 \frac{Q^2}{M^2 - m^2 + Q^2} \frac{1}{[(M + m)^2 + Q^2][(M - m)^2 + Q^2]} \times \\ \times \left(1 + \frac{Q^2}{M^2}\right) \left(1 + \frac{M}{m}\right)^2 \left(1 - \frac{M}{m}\right) [G_c^*(Q^2)]^2, \quad (16)$$

where  $M$  is the isobar mass. As follows from (16)  $f_L(0, Q^2)_\Delta$  is much smaller than  $f_L(0, Q^2)_{proton}$  (10) up to  $Q^2 \sim 0.5 - 1 \text{ GeV}^2$ . Therefore one may expect, that in the case of the forward scattering amplitude of longitudinal photon on the proton at zero energy, eq.(10), may represent the whole  $f_L(0, Q^2)$  up to  $Q^2 \approx 0.5\text{GeV}^2$ .

The physical applications of these results will be the subject of a separate publication.

I am thankful to V.M.Belyaev for useful discussions. This work was supported in part by INTAS Grant 93-0283.

- [1] W.Thirring, Phil.Mag. **41**, 1193 (1950).
- [2] N.Kroll and M.Ruderman, Phys.Rev. **93**, 233 (1954).
- [3] F.E.Low, Phys.Rev. **96**, 1428 (1954).
- [4] M.Gell-Mann and M.L.Goldberger, Phys.Rev. **96**, 1433 (1954).
- [5] H.F.Jones and M.D.Scadron, Ann.Phys. **81**, 1 (1973).